Pendulum Walker

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Abstract:
Picture someone walking from left to right. During one step (intra-step) we treat them as a simple pendulum. This model is called the *rimless wheel* in the literature. We analyze this model intra-step using dynamic programming to find the optimum energy profile based on time and energy cost. We then analyze the problem inter-step for the ideal stepsize based on time cost alone, i.e., without foot collision (energy) losses.

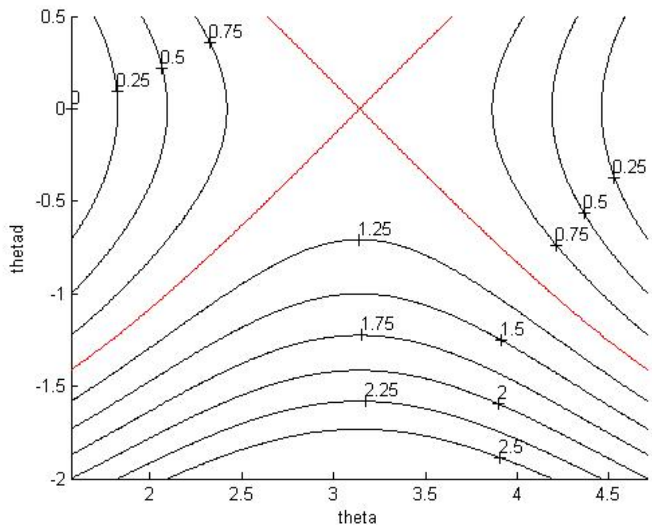

Figure 1: Phase Plane of the Inverted Pendulum

Introduction: Unpowered walking
We derive the pendulum equation as shown in Appendix A (see Figure 9 for coordinate system). From equation (17), divide through by *m* and set $g = l = 1$ and the scalar energy lines of the pendulum as plotted in Figure 1 are given by:

$$E = \frac{1}{2}\dot{\theta}^2 - \cos\theta \qquad (1)$$

A valid *trajectory* for our walker starts on the right-hand side of Figure 1 where $\theta > \pi$ and travels right to left (as opposed to the motion of the pendulum itself, which is clockwise from about 10 to 2 o'clock). If there is enough energy, the pendulum rotates over the top until foot collision at some angle, $\theta = a$, before it hits the ground at $\theta = \frac{\pi}{2}$. Energy levels (which are directly related to the relative speed of the pendulum at a given angle) are depicted as contour lines on the above map. We ignore the angles where the pendulum is below the ground, because obviously, that doesn't mean a whole lot to us. The red line is the separatrix at E=1. Above the red line, the pendulum doesn't have enough energy to rotate over the top and the behavior fundamentally changes to an

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oscillation about the other fixed point at $\theta = 0$ (not shown). For us, this simply means the pendulum falls backwards until it hits the ground.

Powered Walking

Now, we'd like to create an algorithm for calculating the minimum cost in time and energy to operate a powered pendulum given by:

$$\ddot{\theta} + \sin\theta = u \tag{2}$$

Powering the pendulum adds energy to the system; hence it is no longer a conservative system (see Strogatz, p. 170-173). However, we can calculate the change in energy by taking the time derivative of the conservative equation (1) and comparing the result to the powered system in (2):

$$\frac{dE}{dt} = \frac{d}{dt}\left(\frac{1}{2}\dot{\theta}^2 - \cos\theta\right) = \dot{\theta}\left(\ddot{\theta} + \sin\theta\right) = u\dot{\theta}$$

We see from the above equation that at lower speeds, $\dot{\theta}$, a higher torque, *u*, is required for a given energy change, ΔE. For convenience, we will treat the powered system as being intermittently powered but then coasting along the constant energy contours shown. This gives us a starting point for calculating the cost of energy change. We use the simple case where the pendulum will have enough energy to make it over the hump at $\theta = \pi$. (This avoids nasty division by zero problems). Furthermore, we will calculate cost based upon ΔE, not on *u*. This will free us from having to adjust our answers depending on $\dot{\theta}$.

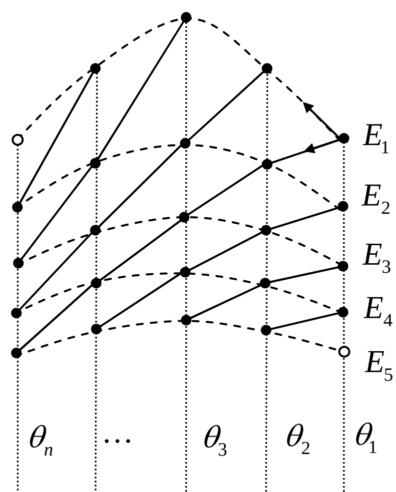

Figure 2: Phase plane of the pendulum walker with dynamic programming

Dynamic Programming

The algorithm we'd like to develop is based upon a dynamic programming example shown in Appendix B. Let's take what we've learned there and apply it to the pendulum walker. Instead of 4 possible choices per node, we designed the problem to only have two choices per node: 1) Stay on the energy level you are currently at--dotted line or 2) Move up to the next level--solid line. Starting from the upper right in Figure 2 and working your way left, you can see how this would look. We do our accounting with the functional equation:

$$f_i = \min_j \left\{ t_{ij} + f_j \right\} \quad \text{for} \quad i \neq end \tag{3}$$

This all comes down to how you number the nodes and build your Cost matrix. As you can see in Figure 2, 3, and 4 there are five Energy levels: $E_1 \ldots E_5$ and *n* increments of θ. Starting from the upper right (Node: 1 in Figure 3) which is at energy level: E_1 and angle: θ_1 (as shown in Figure 3) and moving left you have two choices as depicted by the two arrows. You can head to node six by staying on the same *E*- level, which would be node number *m*+5 in general, where *m* is the current node number. Or you may move to node number seven (*m*+6 in general) by moving "up" to Energy level E_2. If you stay on the same energy contour then your only cost is the time cost Δt to go from θ_n to θ_{n+1}. Now keep in mind that the angle $\Delta\theta$ is a constant, but the time cost, Δt, varies depending on where you are at on the graph. If you move up an energy level, then you incur a cost equal to both Δt and ΔE.

$$Cost = \left| \begin{array}{c} \Delta t \\ \Delta t + \Delta E \end{array} \right. \text{ depending on path taken} \tag{4}$$

this corresponds to an arc *length* as defined in Appendix B:

$$t(i, j)=Cost(m,m+n+4) \tag{5}$$

and the total cost for *n*=1:2 is

$$Total\_Cost(m,n)=Cost(m,n)+f(m+n+4); \tag{6}$$

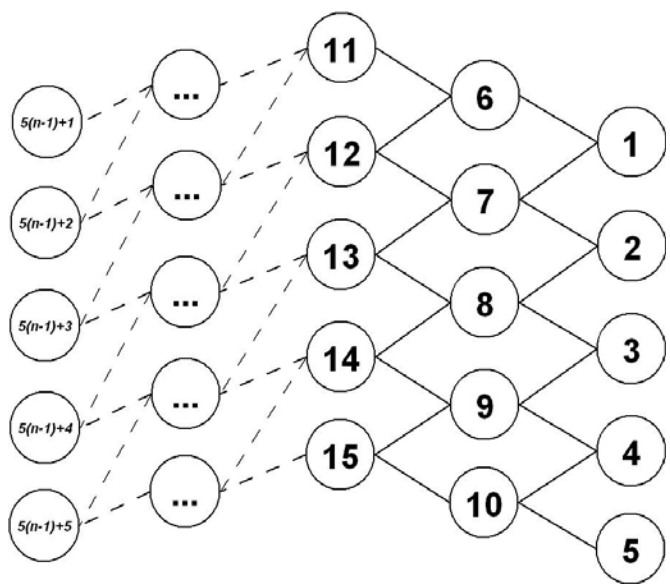

Figure 3: Node Numbering Scheme

We then proceed as shown in the appendix, and when we are done we create a decision tree that shows what the optimal path should be for different starting positions in phase space. Generally we will start at the upper right hand side of the graph and move right to left. Let's say, for example, that you started at the lowest energy level, *E*=1.01 (shown as the top curve of Figure 4). This energy level is barely beyond the separatrix at E=1, and this means that you will make it over the top, albeit *very* slowly. The graph shows that

you should move "up" one level (move down on the graph) to E=1.126. Basically, the graph is telling us what should be intuitively obvious: at low energy levels, it will take so long to get "over the hump" to the other side, you might as well spend the energy and gain the time savings of being on a higher energy level (read: higher speed at a given angle) and furthermore, you should do this as quickly as possible. What might not be as obvious is that this logic has limits: there comes a point at which it is no longer worth it to expend the energy to reach a new E-level, i.e., higher, speed. Look at the E=1.01 energy curve as we move over the hump to the left. There comes a point (around theta=2.8) where it is advisable to stay on this same energy curve. You could think of this with the logic that if you're already over the hump, what's the use in spending the additional energy now? You've already come so far!

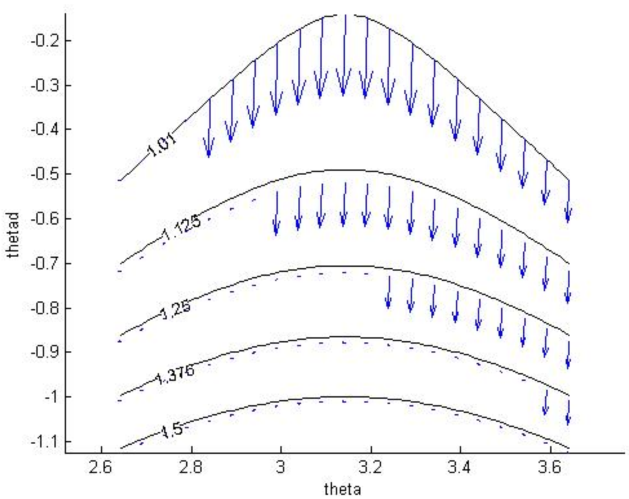

Figure 4: Dynamic Programming Results

Optimum Stepwidth without Collision Loss or Best Angle

Let's assume, for a moment, that you could create a walking vehicle that had no collision loss. We could argue all day until we're blue in the face about what device could possibly do this, but the bottom line is this: what would be the fastest (average forward velocity) step size if we could? This provides a rock-bottom energy analysis that should prove useful as a guide for building practical walking systems.

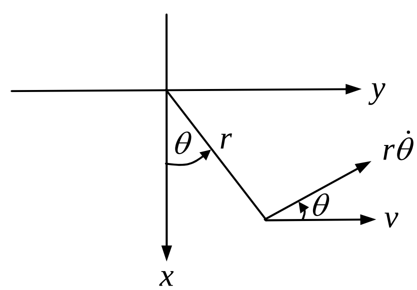

Figure 5: Pendulum Walker

To solve the problem, we must find the average velocity, $\bar{v}$, but first we need to find v. Looking at Figure 5, we see the familiar transformation from Cartesian to cylindrical

coordinates[2]. The velocity, v, we are interested is along the y-direction and is given by $v = r\dot{\theta}\cos\theta$ so if r=1, then

$$v = \dot{\theta}\cos\theta \tag{7}$$

We know from past work that (recall that $\dot{\theta}$ is negative):

$$\dot{\theta} = -\sqrt{2}\sqrt{E + \cos\theta} \tag{8}$$

Substituting (8) into (7), we obtain:

$$v = -\sqrt{2}\sqrt{E + \cos\theta}\cos\theta \tag{9}$$

We can see a graph of this function in Figure 6. Basically, we get a cosine penalty as we approach $\frac{3\pi}{2}$ from the left or $\frac{\pi}{2}$ from the right. This corresponds to the situation in Figure 5 where we are going to hit the ground to the left or to the right. At those times you are moving in an up and down direction and not left to right (it's like when you're doing the splits), so it's clear that you would like to be centered around the vertical: $\theta = \pi$, where the cosine penalty is minimal (most of your rotational motion translates to forward motion here). What's not clear from the pictures is why the linear velocity, v, has a minimum at $\theta = \pi$. The reason for that is $\dot{\theta}$ has a minimum at $\theta = \pi$. Why? Well, you could look at a graph of equation (8), or you could use common sense to think that gravity is working against the pendulum as it approaches its peak and its Potential Energy (which is proportional to height: $\cos(\theta)$) is maximum there, and since the pendulum is a conservative system, its Kinetic Energy (which is proportional to $\dot{\theta}^2$) must be minimum there because the overall energy must remain constant. This creates an interesting scenario: at low energies we get a situation like you see in Figure 6. This is a graph of v with E=1.01, so the pendulum is barely above the separatrix. This means that it slows down a whole lot near the peak.

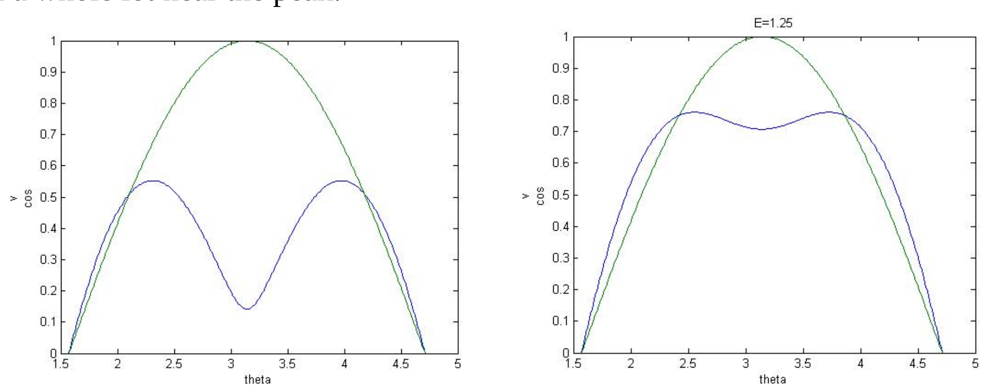

Figure 6, Linear velocity, v, as a function of θ (in blue) plotted with cosine (in green)

With a situation like we see in Figure 6, we might as well take a step, because if we are not losing anything at collision it makes sense to increase our average speed by taking in the dual peaks of the velocity curve, v, to the left and right of $\theta = \pi$. At higher values of E as shown in Figure 6(b), we start to see the minimum at $\theta = \pi$ become less pronounced until finally, in Figure 7 at E=1.5 the minimum has become a maximum.

[2] Never mind that the actual position of the walker is in the upper half plane: the math will take care of that.

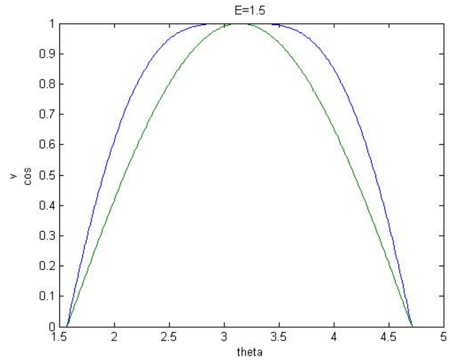

Figure 7: Linear velocity, *v*, as a function of θ at *E*=1.5

Now, we shouldn't confuse anyone, $\left|\dot{\theta}\right|$ always has a minimum at $\theta = \pi$, but we're plotting, *v*, which has the cosine penalty tacked on multiplicatively. What's happening is that at higher and higher energies, the change in potential energy due to the pendulum swinging around is relatively small compared to the total energy. This means that E, which is made up mostly of kinetic energy now, is not affected much by changes in θ, which means $\dot{\theta}$ is then a relatively flat (approximately constant) function with respect to θ. This causes $v \approx -\cos\theta$, as plotted in Figure 7, for $\theta \approx \pi$ and *E*=1.5 in equation (9). Thus the minimum at $\theta = \pi$ has disappeared entirely, indeed it is now a *maximum* and there is no point in taking a step further, you might as well run or roll.

Heuristically now, you should have a sense of what happens. We are left with a couple questions: First, how do we find out what is the best step size for a given energy level, *E*? Secondly, how do we calculate the maximum energy (seen in the graphs as occurring at *E*=1.5) where it makes sense to walk?

Optimal Step Size

The first question is best answered by taking a look at how we calculate averages: Recall:

$$\overline{v}_n = \frac{1}{n}\left(v_1 + v_2 + \ldots v_n\right) \tag{10}$$

where *n* is the number of samples taken. How would you know if taking next bigger stepsize is better? We assume that the stepsize is centered around $\theta = \pi$, and we iterate so that the stepsize is incrementally bigger. Thus if

$$\overline{v}_{n+1} > \overline{v}_n \tag{11}$$

we know that a bigger stepsize has a better average linear velocity. From Figure 2, we see that there will come a point where it is no longer better to take a bigger step. From (10) we see that

$$\bar{v}_{n+1} = \frac{1}{n+1}\sum_{1}^{n+1} v_i$$
$$= \frac{1}{n+1}\sum_{1}^{n} v_i + \frac{1}{n+1} v_{n+1}$$
$$= \frac{1}{n+1}\bar{v}_n + \frac{1}{n+1} v_{n+1}$$

so if we substitute $\bar{v}_{n+1}$ from above into (11) we get:

$$\frac{n}{n+1}\bar{v}_n + \frac{1}{n+1} v_{n+1} > \bar{v}_n$$

and re-arranging terms, we find

$$\frac{n-(n+1)}{n+1}\bar{v}_n > -\frac{1}{n+1} v_{n+1}$$

finally, the condition for satisfying (11) is

$$v_{n+1} > \bar{v}_n \qquad (12)$$

In other words, if the next incremental velocity is greater than your average, take it.

This can be seen graphically by looking at the average as we increase the stepsize. Figure 8 contains the right half of the graph of linear velocity, *v*, as seen in Figure 6 through Figure 7. Since those graphs are symmetric, we can take the average over the right half side, and it will be the same as the average over the whole curve. So we see that the average velocity increases until it reaches the condition in equation (12) where the curves cross, and from then on the average velocity decreases.

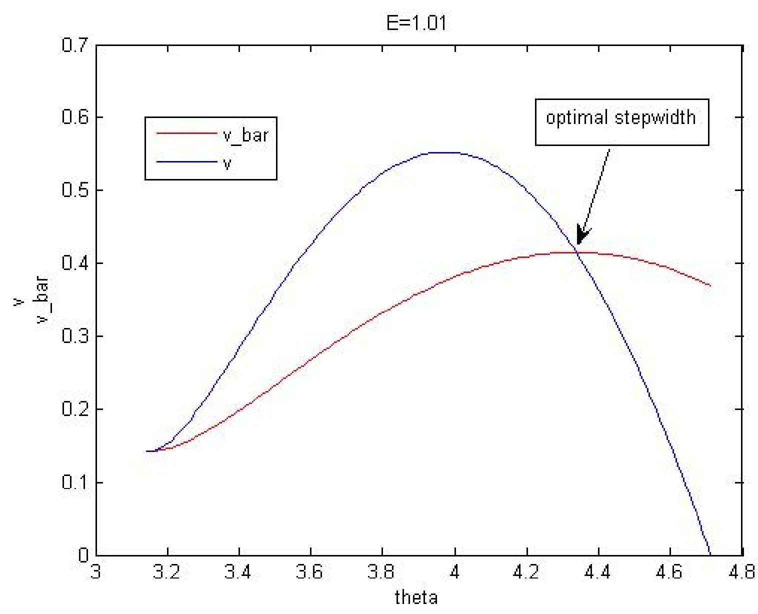

Figure 8: Optimum Stepwidth at E=1.01 is about 138^0

The optimum stepwidth is calculated by taking the angle between where the curves cross, θ_{cross}, and π and multiplying that quantity by two (by symmetry with the left hand side of the curve).

$$\theta_{best} = 2\left(\theta_{cross} - \pi\right) \tag{13}$$

The same trend we saw in Figures 6 through 7 continues here, at higher energy levels, the optimum stepwidth gets narrower and narrower, until at *E*=1.5, it makes no sense to take a step at all (stepwidth approaches zero).

Maximum Walking Energy

We've seen clues graphically that E=1.5 is the maximum energy where it makes sense to walk. Is there another way we can find this value? We seek an answer to the second question we posed above. We start from the equation for the forward velocity, equation (9), which we repeat here:

$$v = -\sqrt{2}\sqrt{E + \cos\theta}\,\cos\theta$$

If we examine the graphs in Figures 6 and 7 above, we see that the minimum at $\theta = \pi$ becomes a maximum as we change energy. If we take the derivative of *v* with respect to θ, we see that

$$\frac{dv}{d\theta} = \frac{\sqrt{2}}{2}\cos\theta\frac{\sin\theta}{\sqrt{E+\cos\theta}} + \sqrt{2}\sin\theta\sqrt{E+\cos\theta}$$

Simplifying:

$$\frac{dv}{d\theta} = \frac{\sqrt{2}}{2}\sin\theta\left(\frac{2E + 3\cos\theta}{\sqrt{E+\cos\theta}}\right) \tag{14}$$

You can see that at $\theta = \pi$ there is always extremum, because $\sin\pi = 0$. We can now examine the sign of $\frac{dv}{d\theta}$ to test if *v* has a minimum or a maximum at $\theta = \pi$. Recall that if $\frac{dv}{d\theta}$ changes from + to – at $\theta = \pi$, then *v* has a local maximum there. Recall that $\sin\theta$ changes from + to – through $\theta = \pi$. So what happens there depends on the value of *E* in the numerator and denominator of the term in brackets. Clearly, $E > 1$ so that nothing strange happens under the square root. Now we are left with the numerator: If the numerator, $2E + \cos\theta$, is positive then the sign of $\frac{dv}{d\theta}$ changes from + to – and we have a local maximum there. The condition for this local maximum is:

$$2E + 3\cos\pi > 0$$

simplifying:

$$E > \frac{3}{2} \tag{15}$$

which is just what we saw graphically: above *E*=1.5, it makes no sense to take a step.

We can check this result with the second derivative test:

$$\frac{d^2v}{d\theta^2} = \frac{\sqrt{2}}{2}\cos\theta\left(\frac{2E+3\cos\theta}{\sqrt{E+\cos\theta}}\right) + \frac{\sqrt{2}}{2}\sin\theta\left(\frac{-3\sin\theta}{\sqrt{E+\cos\theta}} + \frac{1}{2}\sin\theta\frac{\left(2E+\cos\theta\right)}{\left(E+\cos\theta\right)^{\frac{3}{2}}}\right) \tag{16}$$

We can see immediately that the whole second term goes to zero at $\theta = \pi$ because $\sin \pi = 0$. This means that the inflection point is again where the numerator of the first term equals zero, i.e., $2E + \cos\theta = 0$. This is, of course, at E=1.5.

Appendix A: The Simple Pendulum

Cough, cough, do you smell the dust? This is a hoary problem, but it is the basic dynamics of our walker. Let's start by deriving the equation of motion using the conservation of energy.[3] We assume a rigid, massless rod.

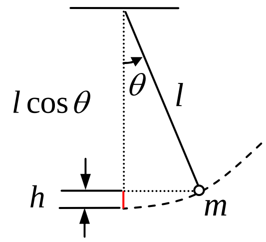

Figure 9: A simple pendulum

First of all the speed of the the pendulum bob is given by:

$$v = l\dot{\theta}$$

The kinetic energy of the bob is then:

$$T = \frac{1}{2}mv^2 = \frac{1}{2}ml^2\dot{\theta}^2$$

We have the potential energy also:

$$V = mgh = mg(l - l\cos\theta)$$

where we have chosen the datum, where the potential energy is zero, to be at the bottom of the pendulum's swing. This is an arbitrary choice. We can just as easily choose the datum to be at the ground level. Then $V = -mgl\cos\theta$ The total energy is then given by:

$$E = T + V = \frac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta \qquad (17)$$

Either way, the conservation of energy says that $\frac{dE}{dt} = 0$, and we obtain:

$$ml^2\dot{\theta}\ddot{\theta} + mgl\dot{\theta}\sin\theta = 0$$

where we used the chain rule. This has two solutions, either $\dot{\theta} = 0$ always, or

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

which is the familiar equation of motion.

Appendix B: Dynamic Programming[4]

[3] This derivation can be found online at http://www.scar.utoronto.ca/~pat/fun/NEWT3D/PDF/

[4] This example is from *Dynamic Programming: Models and Applications* Eric V. Denardo

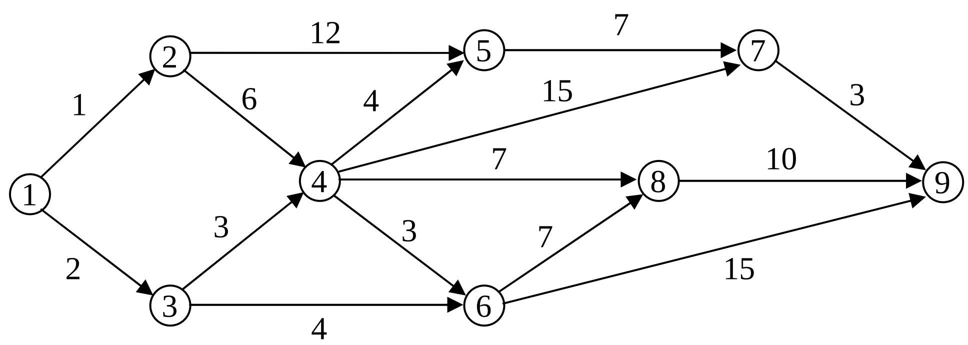

Figure 10: A Directed Network

This directed network example consists of a set of nine nodes and a set of 15 directed arcs between them. The directed arcs are ordered pairs (*i*, *j*) where *i* and *j* are nodes. The *length* of the directed arc (*i*, *j*) is denoted as t_{ij}, for example: $t_{24} = 6$. In this optimization problem, we wish to find the shortest path through the network. Let

$$f_i = \text{the minimum travel time from node } i \text{ to node } 9$$

By definition, $f_9 = 0$, and we interpret

$$t_{ij} + f_j$$

as the travel time of the path from node *i* to node 9 that first traverses arc (*i*, *j*) and then travels as quickly as possible from node *j* to node 9. As this is a path from node *i* to node 9, its travel time must be at least as large as f_i (otherwise f_i is not a minimum path.) and in reality $t_{ij} + f_j$ could be a lot longer than f_i. Thus

$$f_i \le t_{ij} + f_j \quad \text{for} \quad i \ne 9$$

There is a fastest path from node *i* to node 9 and it traverses *some* arc (*i*, *j*) first and then gets from node *j* to node 9 as quickly as possible. So some *j* satisfies the above inequality as an equality, and this is the *functional equation:*

$$f_i = \min_j \left\{ t_{ij} + f_j \right\} \quad \text{for} \quad i \ne 9 \tag{18}$$

The set of *j* over which the right hand side of equation (18) is to be minimized occurs over those *j* for which (*i*, *j*) is an arc (kind of hard to minimize otherwise). Each of those nodes has a corresponding f_j (the min path from *j* to the end) and each of the arcs to node *j* has a corresponding length t_{ij}. One of those *j* happens to be the minimum path, and that path determines the value for f_i. To solve the problem, you *embed* it in the larger problem of finding the minimum path from *every* node. Starting from node $i = 8$, using equation (18) for every node to *backtrack* the answer: (1,3,4,5,7,9).

Algorithm Development

Now the question remains, how do we solve this problem on a computer? Well, first we must put the directed network in Figure 10 into a cost matrix, C, for calculation purposes. We set up the cost matrix like this:

$$C(m,n)=\begin{pmatrix} 1 & 2 & NaN & NaN \\ NaN & 6 & 12 & NaN \\ 3 & NaN & 4 & NaN \\ 4 & 3 & 15 & 7 \\ NaN & 7 & NaN & NaN \\ NaN & 7 & 15 & NaN \\ NaN & 3 & NaN & NaN \\ 10 & NaN & NaN & NaN \end{pmatrix}$$

There are eight rows in the matrix, and the row number *m* is the same as the node number: *i*=*m*. We only need eight rows, because we already know the value for $f_9 = 0$. We could have made a square matrix, so the column number would be the same as *j*, but this would have been a really sparse matrix. Looking at Figure 10, we see that the nodes are labeled so that each arc (*i, j*) has *i* < *j*. This means that $C(i, j)$ would be empty for all $j \le i$. That is why we decided instead to make the column number *n* refer to $j = i + n$. For example, if we are in row 2 (node 2) the third column is the directed arc (2, 2+3) = (2,5) which has length $t_{2,5} = 12$. We only have to make a total of 4 columns because no node has more than four directed arcs emanating from it. *NaN* is a nonexistent entry, a placeholder which is ignored when calculating the minimum. Thus, to find t_{ij} from $C(m,n)$ we substitute:

$$t(i, j) = C(m, m+n)$$

For computational compatibility, we need to pad *f*. We know that $f_9 = 0$, but we also need $f_{8+4} = f_{8+3} = f_{8+2} = NaN$. This keeps the nonexistent paths nonexistent because: $NaN + \alpha = NaN$. At last we find the total cost $t_{ij} + f_i$ and minimize to find f_i by the functional equation (18):

```
for m = 8:-1:1
        for n = 1:4
                Total_Cost(m,n)=C(m,n)+f(m+n);
        end
        f(m)=min(Total_Cost(m,:));
end
```

We have now built a *Total_Cost* matrix which gives the cost for *each path from* a given node *i*=*m* to the end. We then take the min of that row (the equivalent of taking the min over *j* as shown in the functional equation (18)) to find the minimum path from a given node to the end. When we are done, we not only know the minimum path from the beginning node to the end, we also know the minimum path from every other node to the end. This is a basic property of dynamic programming: you embed the smaller problem of finding the minimum path into the larger problem of finding the minimum path from every node, and you then solve the smaller problem by first solving the larger problem.

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